Assortment Optimization under Variants of the Nested Logit Model

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Assortment Optimization Problem

Input:

- n items
- Revenue for each item: r_i
- Set dependent probability of purchase for each item:

$$P(i \text{ is purchased when } S \text{ offered}) = P_i(S)$$

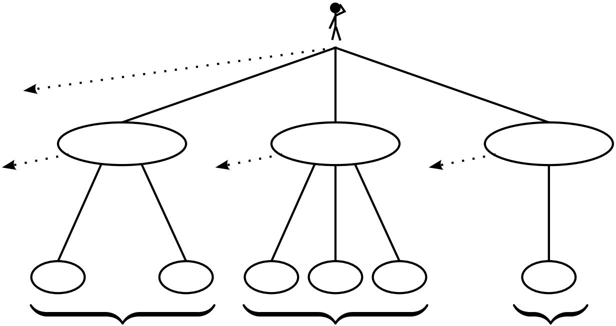
Output:

- Set S that maximizes expected revenue $\sum_{i \in S} r_i P_i(S)$

Nested Logit Model Determines $P_i(S)$

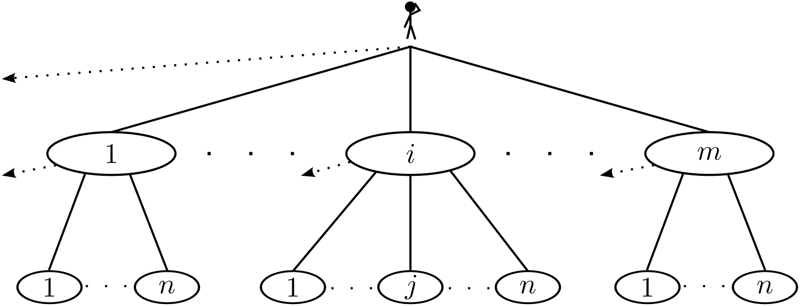
High level idea:

- Customer chooses a nest, or leaves
- Customer purchases an item, or leaves



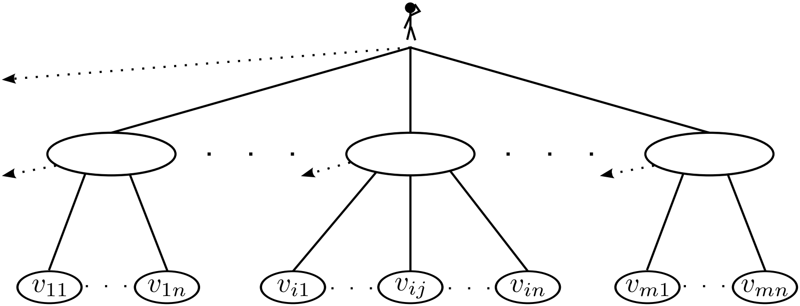
$$S = S_1 \cup S_2 \cup S_3$$

Model Paramaters



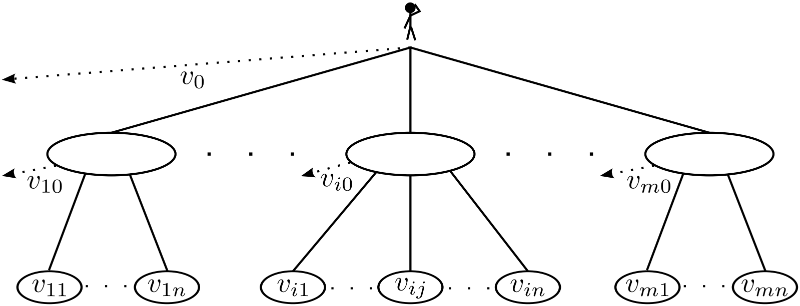
- m nests
- n items

Model Paramaters



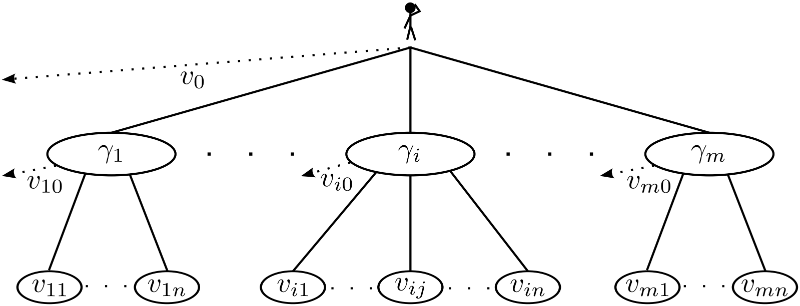
- Item j in nest i has preference weight v_{ij}

Model Paramaters



- Outer no purchase (0 revenue) has weight v_0
- No purchase (0 revenue) for nest i has weight v_{i0}

Model Paramaters



- Dissimilarity parameter γ_i for nest i

$P_i(S)$ in Nested Logit Model

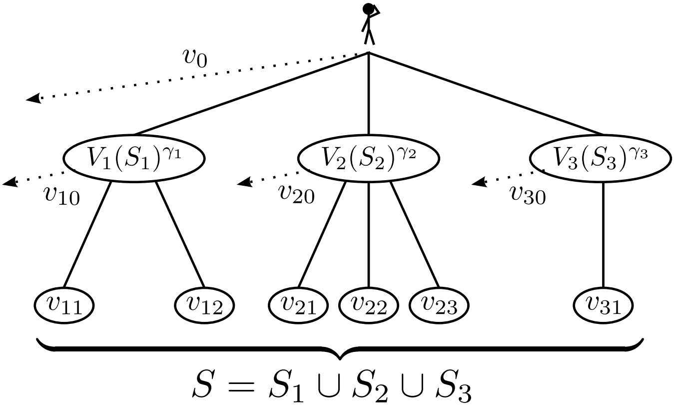
Notation:

- Offer set $S = S_1 \cup \ldots \cup S_m$
- $V_i(S_i) = v_{i0} + \sum_{j \in S_i} v_{ij}$

Probabilities:

- P(selecting nest i) $= \frac{V_i(S_i)^{\gamma_i}}{v_0 + \sum_k V_k(S_k)^{\gamma_k}} \propto V_i(S_i)^{\gamma_i}$
- P(selecting item j | nest i selected) $= \frac{v_{ij}}{V_i(S_i)} \propto v_{ij}$
- P(selecting item j) $= \frac{v_{ij} V_i(S_i)^{\gamma_i - 1}}{v_0 + \sum_k V_k(S_k)^{\gamma_k}}$

$P_i(S)$ in Nested Logit Model



Expected Revenues in Model

Expected revenue given nest i selected:

$$R_i(S_i) = \sum_j r_{ij} \frac{v_{ij}}{v_{i0} + \sum_{k \in S_i} v_{ik}}$$

Expected revenue from $S = S_1 \cup \ldots \cup S_n$:

$$\sum_i R_i(S_i) \frac{V_i(S_i)^{\gamma_i}}{v_0 + \sum_k V_k(S_k)^{\gamma_k}}$$

Full Problem Statement

Input:

- m nests
- n items in each nest
- Revenue for item j in nest i: r_{ij}
  - Assume $r_{i1} \geq \ldots \geq r_{in}$
- Preference weights:
  - Item j in nest i: v_{ij}
  - Outer no purchase: v_0
  - No purchase for nest i: v_{i0}
- Dissimilarity parameter for nest i: γ_i

Output:

- Set $S = S_1 \cup \ldots \cup S_m$ that maximizes expected revenue

$$\sum_i R_i(S_i) \frac{V_i(S_i)^{\gamma_i}}{v_0 + \sum_k V_k(S_k)^{\gamma_k}}$$

Motivation

- Generalization of MNL model, addresses independence of irrelevant alternatives (Ben-Akiva and Lerman 1997)
- Compatible with utility maximization (McFadden 1974, 1981, Borsch-Supan 1990)
- Used in practice (Train et al. 1987, 1989, Lee 1999, Tiwari and Hasegawa 2004, Yates and Mackay 2006)

v_{i0} Values Change Behavior of Problem

$v_{i0} = 0$ for all i

- Customers can't leave nests
- Problem is more tractable

$v_{i0} > 0$ for some i

- Customers can leave nests
- Problem is less tractable

γ_i Values Change Behavior of Problem

$\gamma_i \leq 1$ for all i

- Products in the same nest compete with each other
- Problem is more tractable

$\gamma_i > 1$ for some i

- Products in the same nest synergize with each other
- Problem is less tractable

NP-hardness Depends on v_{i0} and γ_i

- NP-hardness reductions based on subset-sum
- Pseudo-poly-time algorithm exists for all variants

	$\gamma_i \leq 1$	$\gamma_i > 1$
$v_{i0} = 0$	Poly-time solvable	NP-hard
$v_{i0} > 0$	NP-hard	NP-hard

Approximability Depends on v_{i0} and γ_i

Notation:

- ρ is ratio between largest and smallest r_{ij} in any nest
- κ is ratio between largest and smallest v_{ij} in a nest
- $\bar{\gamma}$ is largest γ_i

	$\gamma_i \leq 1$	$\gamma_i > 1$
$v_{i0} = 0$	Poly-time solvable	$\max\{\rho, 2\kappa\}$-appx
$v_{i0} > 0$	2-appx, FPTAS	2κ-appx, PTAS for fixed $\bar{\gamma}$

Poly-time solvable when $v_{i0} = 0$, $\gamma_i \leq 1$

S_i is Revenue Ordered

If $j \in S_i$ all items with revenue $\geq r_j$ are in S_i

Theorem 1

If $S = S_1 \cup \ldots \cup S_n$ is the optimal assortment then S_i is revenue ordered for all i when $v_{i0} = 0$ and $\gamma_i \leq 1$ for all nests i.

Theorem 2

There is a poly-time algorithm to find an optimal assortment when $v_{i0} = 0$ and $\gamma_i \leq 1$ for all nests i.

Theorem 2 does not follow directly from Theorem 1!

Problem Can be Written as Program

$$\min x$$

$$x \geq \max_S \{ \sum_i R_i(S_i) \frac{V_i(S_i)^{\gamma_i}}{v_0 + \sum_k V_k(S_k)^{\gamma_k}} \}$$

$$\min x$$

$$x \geq \sum_i R_i(S_i) \frac{V_i(S_i)^{\gamma_i}}{v_0 + \sum_k V_k(S_k)^{\gamma_k}} \qquad \forall S$$

Constraint for S can be Manipulated

$$x \geq \sum_i R_i(S_i) \frac{V_i(S_i)^{\gamma_i}}{v_0 + \sum_k V_k(S_k)^{\gamma_k}}$$

$$x(v_0 + \sum_k V_k(S_k)^{\gamma_k}) \geq \sum_i R_i(S_i)\, V_i(S_i)^{\gamma_i}$$

$$xv_0 \geq \sum_i R_i(S_i)\, V_i(S_i)^{\gamma_i} - x \sum_k V_k(S_k)^{\gamma_k}$$

$$xv_0 \geq \sum_i V_i(S_i)^{\gamma_i}(R_i(S_i) - x)$$

LP Decomposes by Nests

$$\min x$$
$$xv_0 \geq \sum_i V_i(S_i)^{\gamma_i}(R_i(S_i) - x) \qquad \forall S$$

$$\min x$$
$$xv_0 \geq \sum_i y_{S_i}$$
$$y_{S_i} \geq V_i(S_i)^{\gamma_i}(R_i(S_i) - x) \qquad \forall i \forall S_i$$

Poly-time solvable when $v_{i0} = 0$, $\gamma_i \leq 1$

Theorem 1

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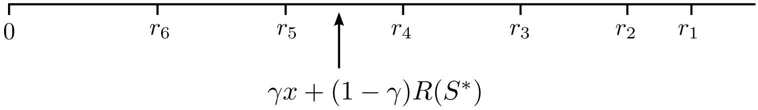
Proof.

- There are at most n configurations for each nest
- There are at most nm constraints in the LP

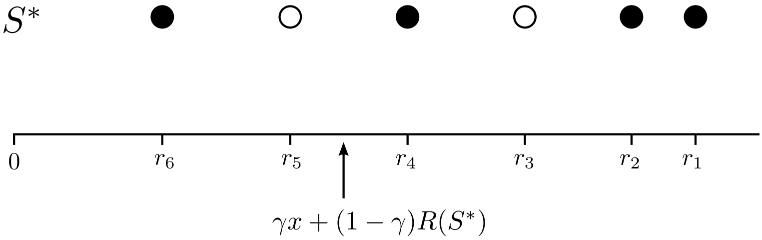
□

Revenue Threshold Exists



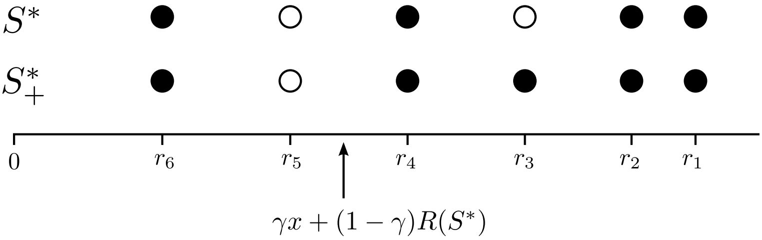
- S^* is optimal for nest, x is total optimal revenue
- Items with revenue above threshold are in optimal solution

Proof by Contradiction



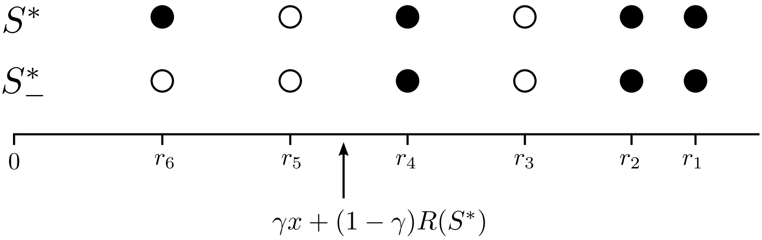
- Suppose S^* is not revenue ordered
- $V(S_+^*)^\gamma(R(S_+^*) - x) \geq V(S^*)^\gamma(R(S^*) - x)$
- $V(S_-^*)^\gamma(R(S_-^*) - x) \geq V(S^*)^\gamma(R(S^*) - x)$

Proof by Contradiction



- Suppose S^* is not revenue ordered
- $V(S^*_+)^\gamma (R(S^*_+) - x) \geq V(S^*)^\gamma (R(S^*) - x)$
- $V(S^*_-)^\gamma (R(S^*_-) - x) \geq V(S^*)^\gamma (R(S^*) - x)$

Proof by Contradiction



- Suppose S^* is not revenue ordered
- $V(S^*_+)^\gamma(R(S^*_+) - x) \geq V(S^*)^\gamma(R(S^*) - x)$
- $V(S^*_-)^\gamma(R(S^*_-) - x) \geq V(S^*)^\gamma(R(S^*) - x)$

$V(S^*_+)^\gamma(R(S^*_+) - x) \geq V(S^*)^\gamma(R(S^*) - x)$

$$V(S^*_+)^\gamma(R(S^*_+) - x) = V(S^*_+)^\gamma\left(\frac{\sum_{j\in S^*_+} v_j r_j}{V(S^*_+)} - x\right)$$

$$= \frac{\sum_{j\in S^*_+} v_j r_j - V(S^*_+)x}{V(S^*_+)^{1-\gamma}}$$

$$= \frac{\sum_{j\in S^*_+} v_j r_j - V(S^*_+)x + v_3(r_3 - x)}{V(S^*_+)^{1-\gamma}}$$

$$\geq \frac{\sum_{j\in S^*_+} v_j r_j - V(S^*_+)x + v_3(1-\gamma)(R(S^*) - x)}{V(S^*_+)^{1-\gamma}}$$

$V(S_+^*)^\gamma(R(S_+^*) - x) \geq V(S^*)^\gamma(R(S^*) - x)$

$$= \frac{\sum_{j \in S_+^*} v_j r_j - V(S_+^*)x + v_3(1-\gamma)(R(S^*) - x)}{V(S_+^*)^{1-\gamma}}$$

$$= \frac{V(S^*)\left(\frac{\sum_{j \in S_+^*} v_j r_j}{V(S^*)} - x\right) + v_3(1-\gamma)(R(S^*) - x)}{V(S_+^*)^{1-\gamma}}$$

$$= \frac{(V(S^*) + (1-\gamma)v_3)(R(S^*) - x)}{V(S_+^*)^{1-\gamma}}$$

$$= \frac{(V(S^*) + (1-\gamma)v_3)(R(S^*) - x)}{V(S^*)^{1-\gamma} + (1-\gamma)V(S^*)^{-\gamma}(V(S_+^*) - V(S^*))}$$

$V(S_+^*)^\gamma (R(S_+^*) - x) \geq V(S^*)^\gamma (R(S^*) - x)$

$$= \frac{(V(S^*) + (1-\gamma) v_3)(R(S^*) - x)}{V(S^*)^{1-\gamma} + (1-\gamma) V(S^*)^{-\gamma} (V(S_+^*) - V(S^*))}$$

$$= \frac{(V(S^*) + (1-\gamma) v_3)(R(S^*) - x)}{V(S^*)^{1-\gamma} + (1-\gamma) V(S^*)^{-\gamma} v_3}$$

$$= \frac{(V(S^*) + (1-\gamma) v_3)(R(S^*) - x)}{V(S^*)^{-\gamma} (V(S^*) + (1-\gamma) v_3)}$$

$$= V(S^*)^\gamma (R(S^*) - x)$$

Computational Results

- $m = 5, n = 25$
- ϵ controls v_{ij} and r_{ij} gaps (smaller ϵ yields larger gaps)
- γ^L is smallest γ_i, γ^U is largest γ_i

Param.	Rev. Ordered		Val and Rev. Ordered	
$([\gamma^L, \gamma^U], \epsilon)$	Avg. Gap	Larg. Gap	Avg. Gap	Larg. Gap
$([0.5, 1.5], 0.6)$	0.01	0.34	0.01	0.20
$([0.5, 1.5], 0.5)$	0.02	0.38	0.02	0.31
$([0.5, 1.5], 0.4)$	0.03	0.66	0.03	0.48
$([0.5, 1.5], 0.3)$	0.05	0.81	0.04	0.78
$([1.0, 2.0], 0.6)$	0.03	0.52	0.03	0.51
$([1.0, 2.0], 0.5)$	0.05	0.71	0.04	0.71
$([1.0, 2.0], 0.4)$	0.08	1.43	0.07	0.92
$([1.0, 2.0], 0.3)$	0.13	2.70	0.11	1.31
$([1.5, 2.5], 0.6)$	0.06	1.56	0.05	0.79
$([1.5, 2.5], 0.5)$	0.09	1.63	0.08	1.03
$([1.5, 2.5], 0.4)$	0.14	2.13	0.12	2.13
$([1.5, 3.0], 0.3)$	0.25	4.21	0.19	2.42
$([2.0, 3.0], 0.6)$	0.09	2.00	0.08	1.24
$([2.0, 3.0], 0.5)$	0.14	2.47	0.12	1.82
$([2.0, 3.0], 0.4)$	0.22	3.70	0.18	2.20
$([2.0, 3.0], 0.3)$	0.38	5.45	0.29	3.26

Open Questions

- FPTAS for general case
- Remove dependence on LP
- Extend to cross nested logit model

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- Remove dependence on LP
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Thank You!