Approximating Graphic TSP by Matchings

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Tobias Mömke and Ola Svensson

2011

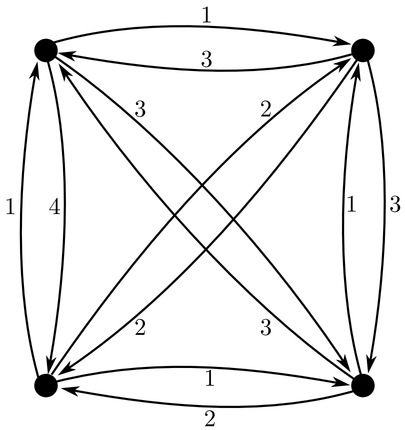
Traveling Salesman Problem (TSP)

Input:

- Complete directed graph $G = (V, A)$
- Arc costs $c_a \geq 0$

Goal:

- Visit each vertex exactly once
- Minimize cost $= \sum_{a \in \text{ tour}} c_a$



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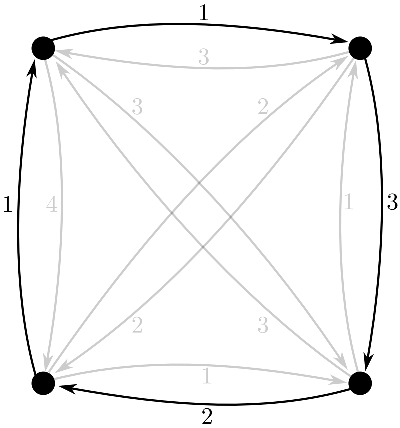


Hardness of TSP

TSP is NP-hard.

Definition: α-Approximation Algorithm

An algorithm that returns a solution of cost no more than $\alpha \cdot OPT$ for every instance, where OPT is the optimal solution cost for the instance.

For any $\alpha > 1$ there is no α-approximation algorithm for TSP.

Special Cases of TSP

Metric: c_e's satisfy the triangle inequality

- $O(\log(n))$-approximation algorithm (Frieze et al., 1982)
- $O\left(\frac{\log(n)}{\log(\log(n))}\right)$-approximation algorithm (Asadpour et al., 2010)

Symmetric: $c_{(u,v)} = c_{(v,u)}$ for all v, u pairs

- $\frac{3}{2}$-approximation algorithm (Christofides, 1976)

Planar: points embedded in Euclidean plane

- PTAS (Arora, 1998) (Mitchell, 1999)

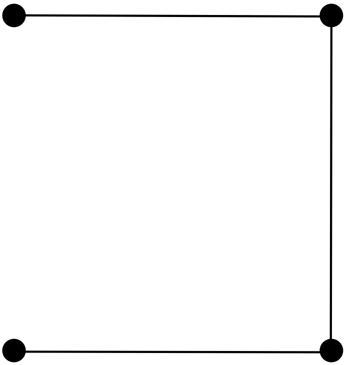
Graph TSP

Input:

- Undirected graph $G = (V, E)$

Create TSP Instance:

- Add edges E' to complete G
- $\forall e \in E \; c_e = 1$
- $\forall e \in E' \; c_e$ is shortest path distance
- Bi-direct all $e \in E \cup E'$



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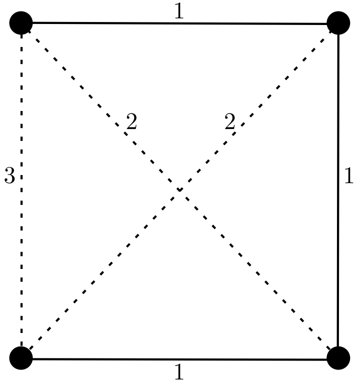

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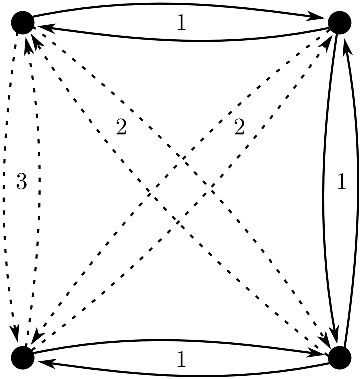
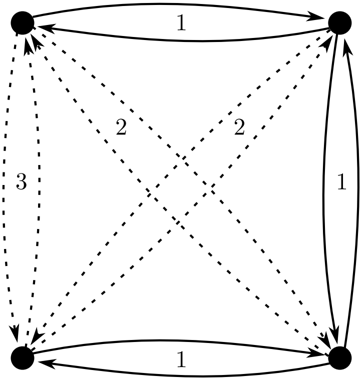


Graph TSP

Properties:

- Valid TSP instance
- c_e's are symmetric
- c_e's are metric



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Graph TSP

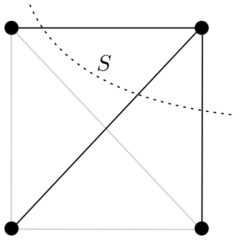
- $\left(\frac{3}{2} - \epsilon\right)$-approximation algorithm (Gharan et al., 2011)

Planar: points embedded in Euclidean plane

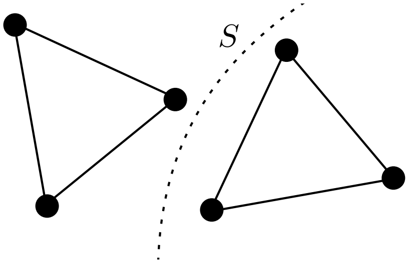
- PTAS (Arora, 1998) (Mitchell, 1999)

Held-Karp LP (Sub-Tour Elimination)

- $\delta(S)$ is the set of edges across S
- $G = (V, E)$ is the TSP input
- x_e variable for all $e \in E$
- Interpretation: $x_e = 1$ iff e in TSP tour



$$\begin{aligned} \min \sum_{e \in E} c_e x_e & \\ \text{s.t.} \quad \sum_{e \in \delta(S)} x_e \geq 2 & \quad \forall \emptyset \neq S \subseteq V \\ \sum_{e \in \delta(v)} x_e = 2 & \quad \forall v \in V \\ x_e \in \{0, 1\} & \end{aligned}$$



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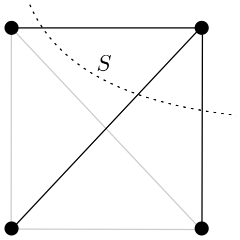

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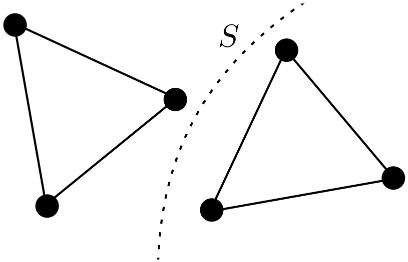
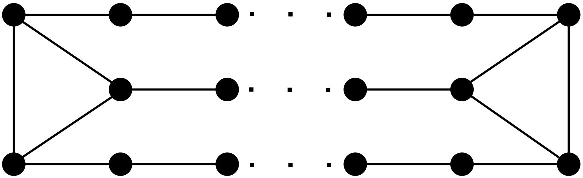


Held-Karp Tight Example

Lower Bound on Integrality Gap

There exists some graph TSP input G where $\frac{OPT(G)}{OPT_{LP}(G)} \geq \frac{4}{3}$.

length k paths



$OPT(G) \approx 4k$
$OPT_{LP}(G) \approx 3k$

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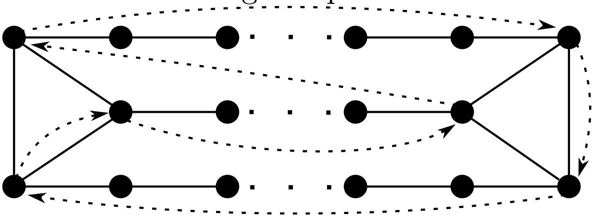

$OPT(G) \approx 4k$
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Held-Karp Conjecture

Integrality Gap (Conjecture)

For all metric symmetric TSP input G, $\frac{OPT(G)}{OPT_{LP}(G)} \leq \frac{4}{3}$.

- $\frac{OPT(G)}{OPT_{LP}(G)} \leq \frac{3}{2}$ for metric symmetric
- $\frac{OPT(G)}{OPT_{LP}(G)} \leq \frac{7}{5}$ for graph TSP on degree 3 bounded graphs

Why would this be good?

Suppose algorithm A rounds LP soln. to "closest" integral soln.

- "closest" soln. has cost at most $\frac{4}{3} OPT_{LP}(G)$
- A's soln. cost is at most $\frac{4}{3} OPT_{LP}(G) \leq \frac{4}{3} OPT(G)$

Approximating Graphic TSP by Matchings

Integrality Gap

For all degree 3 bounded graph TSP input G, $\frac{OPT(G)}{OPT_{LP}(G)} \leq \frac{4}{3}$.

Approximation Algorithm

There is a $\frac{4}{3}$-approximation algorithm for graph TSP on degree 3 bounded graphs.

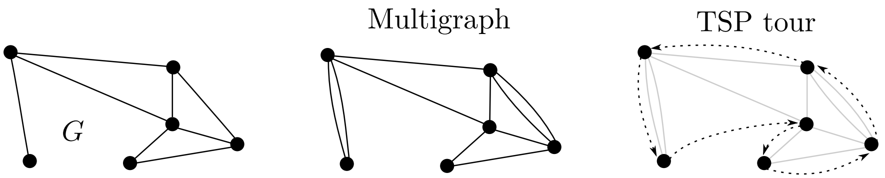
Approximation Algorithm

There is a $\frac{14(\sqrt{2}-1)}{12\sqrt{2}-13} \approx 1.461$-approximation algorithm for graph TSP.

Preliminaries: Graph TSP and Eulerian Multigraphs

- $G = (V, E)$ is original graph TSP input
- $G' = (V, A \cup A')$ is TSP input

Finding a min-cost TSP tour in G' is equivalent to finding a min-cost Eulerian multigraph in G.



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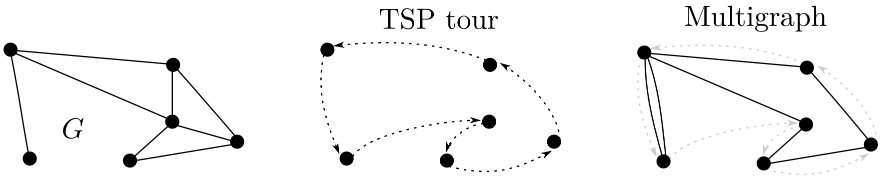
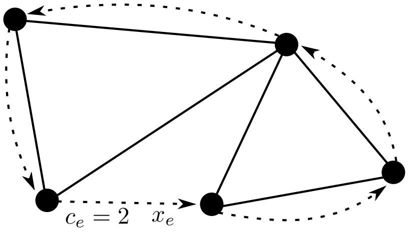


Preliminaries: Held-Karp LP for Graph TSP

- $G = (V, E \cup E')$ is TSP input.
- $x_e \quad \forall e \in E \cup E'$.

$$\begin{aligned} \min \sum_{e \in E \cup E'} c_e x_e & \\ \text{s.t.} \sum_{e \in \delta(S)} x_e \geq 2 & \quad \forall S \subseteq V \\ \sum_{e \in \delta(v)} x_e = 2 & \quad \forall v \in V \\ x_e \geq 0 & \end{aligned}$$



Preliminaries: Held-Karp LP for Graph TSP

- $G = (V, E)$ is original input
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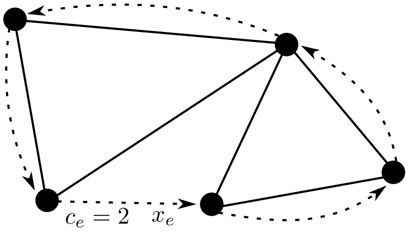

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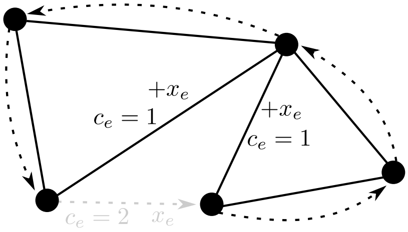


Preliminaries: Held-Karp Cost

The Held-Karp LP has cost at least $n = |V|$ for graph TSP.

$$\min \sum_{e \in E} x_e$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad \forall S \subseteq V$$

$$x_e \geq 0$$

$$\sum_{e \in \delta(v)} x_e \geq 2$$

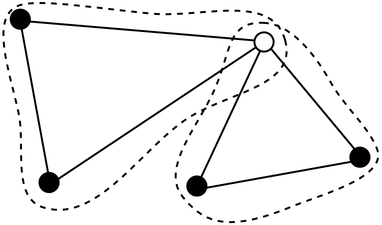
$$\sum_{v \in V} \sum_{e \in \delta(v)} x_e \geq \sum_{v \in V} 2$$

$$2 \sum_{e \in E} x_e \geq 2n$$

$$\sum_{e \in E} x_e \geq n$$

Solving 2-vertex problem solves original:

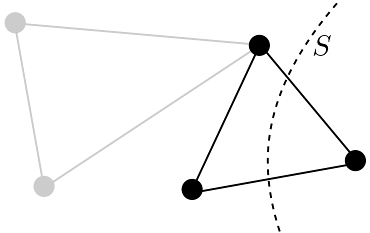
- Decompose (recursively)
- Solve
- Recompose



$OPT_{LP}(G) \geq \sum_{G_S} OPT_{LP}(G_S)$

- Let $\hat{x}_e{}^*$ be optimal for G
- $\hat{x}_e{}^*$ is feasible for sub-problems

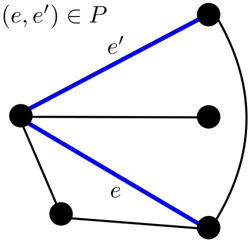
$$\frac{OPT(G)}{OPT_{LP}(G)} \leq \frac{OPT(G)}{\sum_{G_S} OPT_{LP}(G_S)}$$

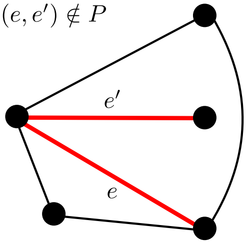


Removable Pairings: Definition

(R, P) is a removable pairing for $G = (V, E)$ if:

- $R \subset E$ (removable edges)
- $P \subseteq R \times R$ (removable pairs)
- A removable edge is in at most one removable pair
- If (e, e') is a pair, e and e' touch a common v of degree at least 3
- After deleting at most one edge in each pair G remains connected





Removable Pairings Lemma

Removable Pairings Lemma

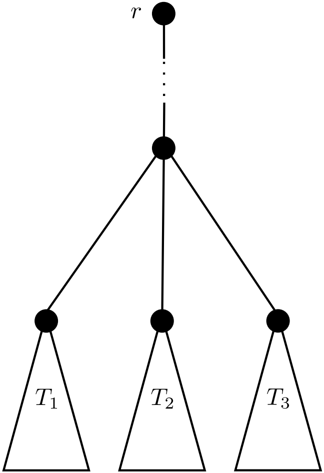
Given a 2-vertex connected graph $G = (V, E)$ and removable pairing (R, P) there is a polytime algorithm that returns a Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

Finding Removable Pairings: Circulation Network

Given $G = (V, E)$ construct a circulation network $C(G, T)$:

- Find a DFS tree T
- Use some gadgets
- Direct $e \in T$ towards leaves
- Direct $e \notin T$ towards root
- $d_e = 1$ for tree arcs
- $d_e = 0$ for non-tree arcs
- $c_e = \infty$ for all e
- Cost of circulation f:

$$\sum_{\text{white } v} \max\{f(In(v)) - 1, 0\}$$



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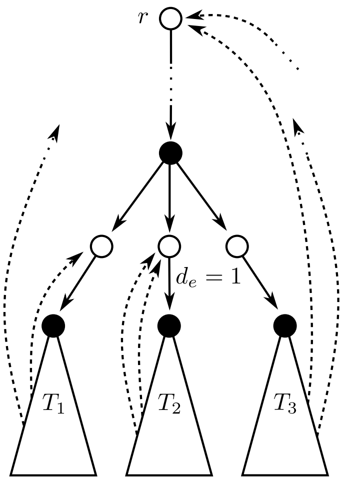


Circulation Lemma

Circulation Lemma

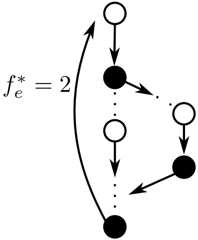
If G is 2-vertex connected there is a spanning Eulerian multigraph in G with cost at most $\frac{4}{3}n + \frac{2}{3}c(f^*) - \frac{2}{3}$.

Notation:

- $C(G, T)$ is circulation network for graph G and DFS tree T
- f^* is optimal circulation for $C(G, T)$
- $c(f^*)$ is cost of f^*
- $f^*(G, T)$ is the support (non-zero arcs) of f^*
- G^* is the result of compressing gadgets in $f^*(G, T)$

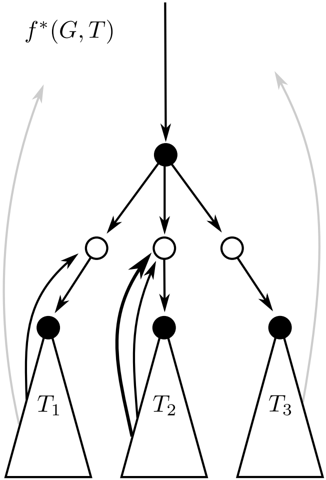
Properties of f^*:

- f^* is integral
- $f^*_e \in \{0,1\}$ on all back arcs



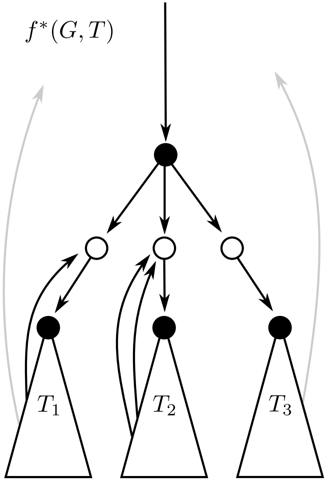
$\sum_{\text{white } v} \max\{f(In(v)) - 1, 0\}$:

- Zero cost arcs
- Non-zero cost arcs



Proof of Circulation Lemma

G^* is 2-vertex connected.



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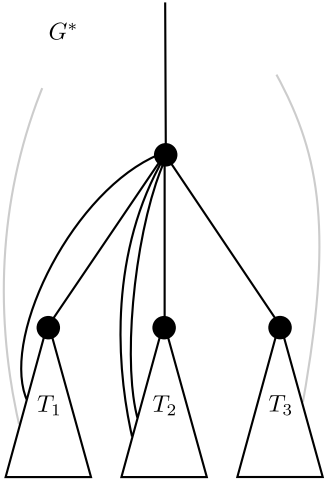


Proof of Circulation Lemma

Create a removable pairing (R_{f^*}, P_{f^*}) on $f^*(G, T)$.

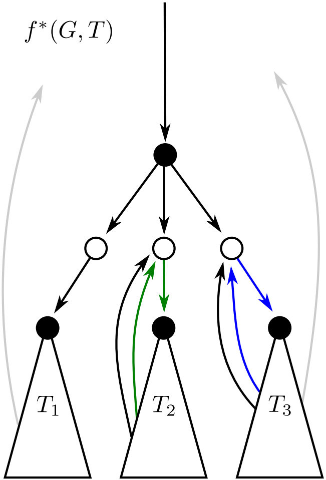
$(e, e') \in P_{f^*}$ iff

- e is a zero-cost back arc
- e' is a tree arc
- e and e' share a common v
- v has 2 incoming arcs

$e \in R_{f^*}$ iff one of the following

- $e \in P_{f^*}$
- e is a back arc

$$|R_{f^*}| - 2|P_{f^*}| \leq c(f^*) + 1$$



Create a removable pairing (R, P) on G^*.

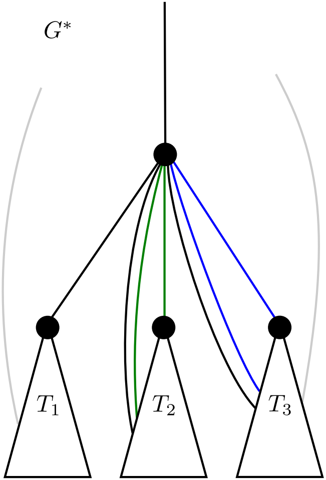
Creation:

- Compress (R_{f^*}, P_{f^*})
- Remove orientation

Properties:

- $|R| = |R_{f^*}|$ and $|P| = |P_{f^*}|$
- $e \in R$ in at most one pair
- If (e, e') touch v, $\delta(v) \geq 3$
- G is connected after removing at most one e from each pair

G^* has $(n-1) + |R| - |P|$ edges



Proof of Circulation Lemma

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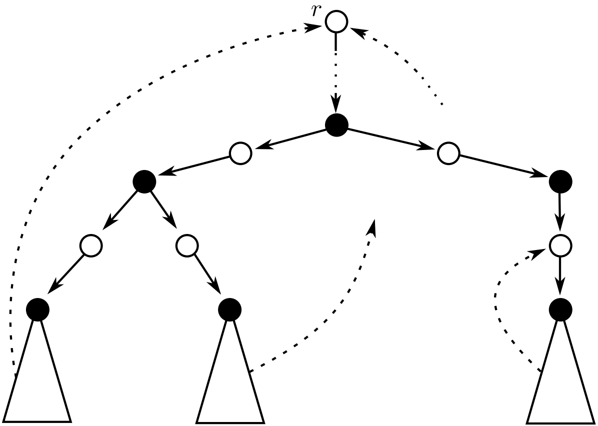
$$\begin{aligned}\frac{4}{3}((n-1) + |R| - |P|) - \frac{2}{3}|R| &= \frac{4}{3}n + \frac{2}{3}(|R| - 2|P|) - \frac{4}{3} \\ &\leq \frac{4}{3}n + \frac{2}{3}c(f^*) - \frac{2}{3}\end{aligned}$$

Circulation Lemma

If G is 2-vertex connected there is a spanning Eulerian multigraph in G with cost at most $\frac{4}{3}n + \frac{2}{3}c(f^*) - \frac{2}{3}$.

Degree 3 Bounded 2-vertex Connected Graphs

There is a circulation of cost at most one in $C(G, T)$.



Degree 3 Bounded 2-vertex Connected Graphs

Circulation Lemma

If G is 2-vertex connected there is a spanning Eulerian multigraph in G with cost at most $\frac{4}{3}n + \frac{2}{3}c(f^*) - \frac{2}{3}$.

- There is a Eulerian multigraph of cost at most $\frac{4}{3}n$
- $OPT(G) \leq \frac{4}{3}n$
- $OPT_{LP}(G)$ (Held-Karp LP cost) is at least n

Integrality Gap

For all degree 3 bounded graph TSP input G, $\frac{OPT(G)}{OPT_{LP}(G)} \leq \frac{4}{3}$.

Degree 3 Bounded 2-vertex Connected Graphs

$\frac{4}{3}$-approximation Algorithm

Input: 2-vertex connected graph $G = (V, E)$

- Solve $LP(G)$. Let $\bar{x_e}$ be the solution vector
- Find a DFS tree T
- Construct circulation network $C(G, T)$
- Solve min-cost circulation on $C(G, T)$ to get (R, P)
- Apply removable pairings lemma

Output: Eulerian multigraph of removable pairings lemma

Cost: $\frac{4}{3}n \leq \frac{4}{3}OPT_{LP}(G)$

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Approximation Algorithm

There is a $\frac{4}{3}$-approximation algorithm for graph TSP on degree 3 bounded graphs.

Approximation Algorithm

There is a $\frac{14(\sqrt{2}-1)}{12\sqrt{2}-13} \approx 1.461$-approximation algorithm for graph TSP.

2-vertex Connected Graphs

Removable Pairing Algorithm

Input: 2-vertex connected graph $G = (V, E)$

- Solve $LP(G)$. Let $\hat{x}_e$ be the solution vector
- Find a DFS tree T but search highest x_e value edge first
- Construct circulation network $C(G, T)$
- Solve min-cost circulation on $C(G, T)$ to get (R, P)
- Apply removable pairings lemma

Output: Eulerian multigraph of removable pairings lemma

Cost: $\frac{4}{3}n + \frac{2}{3}c(f^*) - \frac{2}{3}$

2-vertex Connected Graphs

$$c(f^*) \leq 6(1-\sqrt{2})n + (4\sqrt{2}-3)OPT_{LP}(G)$$

Proof idea:

- Construct a feasible circulation from LP solution.

Removable pairing approximation guarantee is at most:

$$\frac{\frac{4}{3}n + \frac{2}{3}(6(1-\sqrt{2})n + (4\sqrt{2}-3)OPT_{LP}(G))}{OPT_{LP}(G)}$$

Christofides approximation guarantee is at most:

$$\frac{n + OPT_{LP}(G)/2}{OPT_{LP}(G)}$$

Worst case: $OPT_{LP}(G) = \frac{24\sqrt{2}-26}{16\sqrt{2}-15}n$

2-vertex Connected Graphs

$\left(\frac{14(\sqrt{2}-1)}{12\sqrt{2}-13} < 1.461\right)$-approximation Algorithm

Input: 2-vertex connected graph $G = (V, E)$

- Solve $LP(G)$
- If $OPT_{LP}(G) \leq \frac{24\sqrt{2}-26}{16\sqrt{2}-15}n$ run removable pairing algorithm
- If $OPT_{LP}(G) > \frac{24\sqrt{2}-26}{16\sqrt{2}-15}n$ run Christofides algorithm

Output: Eulerian multigraph

Cost: $\left(\frac{14(\sqrt{2}-1)}{12\sqrt{2}-13}\right) OPT_{LP}(G)$

Graph TSPP

Traveling salesman path problem:

- Identical to TSP except forced to start at s and end at t
- $\frac{5}{3}$-approximation algorithm for graph TSPP (Hoogeveen, 1991)

Graph TSPP Approximation Algorithm

For any $\epsilon > 0$ there is a polynomial time algorithm for graph TSPP with performance gaurantee $3 - \sqrt{2} + \epsilon < 1.586 + \epsilon$.

Thanks!

Thanks!

Removable Pairings Lemma

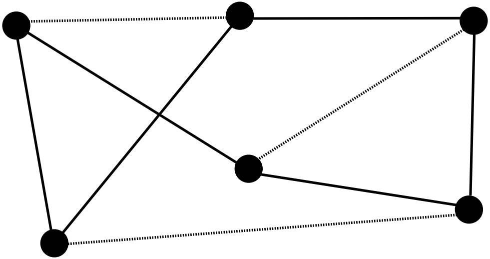
Removable Pairings Lemma

Given a 2-vertex connected graph $G = (V, E)$ and removable pairing (R, P) there is a polytime algorithm that returns a Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

Preliminaries: Matchings

(Almost) Caratheòdory's Theorem

If G is cubic and 2-vertex connected we can, in polynomial time, find a distribution over polynomial many perfect matchings so that any edge is in a matching chosen from this distribution with probability $\frac{1}{3}$.



Extending Carathèodory's Theorem

Extension of Carathèodory's Theorem

Given a 2-vertex connected graph $G = (V, E)$ and removable pairing (R, P) we can, in polynomial time, find a distribution over polynomially many subsets of edges so that a random subset M from this distribution satisfies:

(i) $\forall e \in E$, e is in M with probability $\frac{1}{3}$

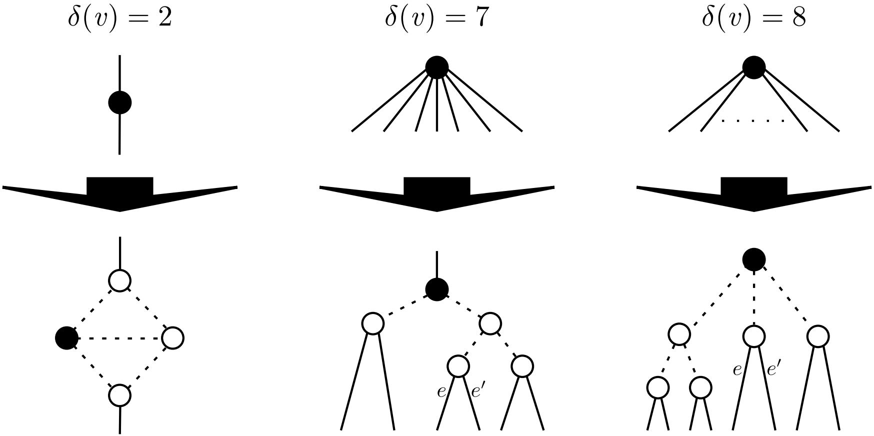
(ii) at most one edge in each removable pair is in M

(iii) $\forall v \in V$ have even degree in the multigraph $(V, E \cup M)$

Proof plan:

- Replace G with a cubic graph G'
- Leverage Carathèodory's Theorem

Proof: Replace G With a Cubic Graph G'



Proof: Leverage Carathèodory's Theorem

Carathèodory's Theorem

... find a distribution over perfect matchings so that any edge is in a matching chosen from this distribution with probability $\frac{1}{3}$.

- Obtain random perfect matching M' for G'
- Restrict M' to G to get the set M

Extension of Carathèodory's Theorem

Given $G = (V, E)$ and removable pairing (R, P) ... M satisfies:

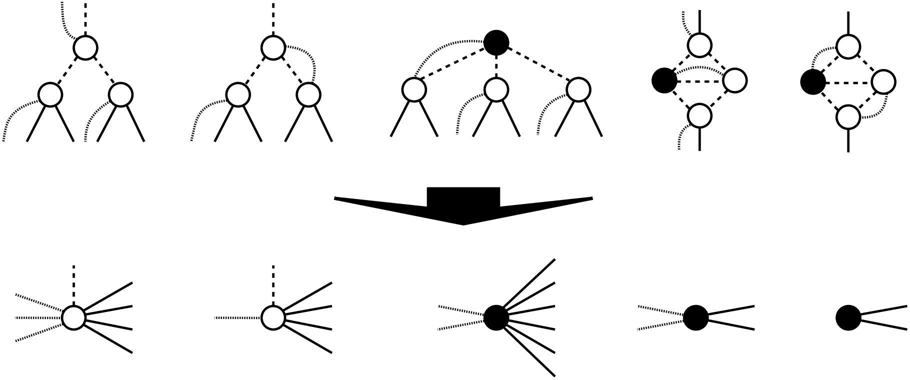
(i) $\forall e \in E$, e is in M with probability $\frac{1}{3}$

(ii) at most one edge in each removable pair is in M

(iii) $\forall v \in V$ have even degree in the multigraph $(V, E \cup M)$

Proof: $(V, E \cup M)$ is Eulerian

- $(V', E' \cup M')$ is a Eulerian multigraph
- Collapsing gadgets retains Eulerian multigraph



Extension of Carathèodory's Theorem

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Proof of Removable Pairings Lemma

Extension of Caratheòdory's Theorem

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(i) $\forall e \in E$, e is in M with probability $\frac{1}{3}$

(ii) at most one edge in each removable pair is in M

(iii) $\forall v \in V$ have even degree in the multigraph $(V, E \cup M)$

- $M = M_R \cup \bar{M}_R$
- M_R is the set of removable edges
- $\delta(v)$ is even for all v in $G' = (V, E/M_R \cup \bar{M}_R)$
- $(V, E/M_R)$ is connected $\rightarrow (V, E/M_R \cup \bar{M}_R)$ is connected

$$\mathbb{E}[|E| - |M_R| + |\bar{M}_R|] = |E| - \frac{1}{3}|R| + \frac{1}{3}(|E| - |R|) = \frac{4}{3}|E| - \frac{2}{3}|R|$$